Linear Convolution

One dimensional linear discrete convolution is defined as:

\[ g(x) = \sum_{s=-\infty}^{\infty} f(s) h(x - s) = f(x) * h(x) \]

For example, consider the convolution of the following two functions:

\[
\begin{array}{c|c|c}
    & f(x) & h(x) \\
\hline
 0 & 1 & 1 \\
 1 & 2 & 1 \\
 2 & 3 & 1 \\
\end{array}
\]

This convolution can be performed graphically by reflecting and shifting \( h(x) \), as shown in Figure 1. The samples of \( f(s) \) and \( h(s - x) \) that line up vertically are multiplied and summed:

\[
\begin{align*}
  g(0) &= f(-1)h(1) + f(0)h(0) = 0 + 1 = 1 \\
  g(1) &= f(0)h(1) + f(1)h(0) = 1 + 3 = 4 \\
  g(2) &= f(1)h(1) + f(2)h(0) = 3 + 2 = 1 \\
  g(3) &= f(2)h(1) + f(3)h(0) = -2 + 1 = -1 \\
  g(4) &= f(3)h(1) + f(4)h(0) = 1 + 0 = 1
\end{align*}
\]

The result of the convolution is as shown below:
Notice that when \( f(x) \) is of length 4, and \( h(x) \) is of length 2, the linear convolution is of length \( 4 + 2 - 1 = 5 \).

## Circular Convolution

One dimensional circular discrete convolution is defined as:

\[
g(x) = \sum_{s=0}^{M-1} f(s) h((x - s) \mod M) = f(x) \ast h(x)
\]

For \( M = 4 \), the convolution can be performed using circular reflection and shifts of \( h(x) \), as shown in Figure 2. The samples of \( f(s) \) and \( h((s - x) \mod M) \) that line up vertically are multiplied and summed:

\[
\begin{align*}
g(0) &= f(3)h(1) + f(0)h(0) = 1 + 1 = 2 \\
g(1) &= f(0)h(1) + f(1)h(0) = 1 + 3 = 4 \\
g(2) &= f(1)h(1) + f(2)h(0) = 3 + 2 = 1 \\
g(3) &= f(2)h(1) + f(3)h(0) = -2 + 1 = -1
\end{align*}
\]

The result of the convolution is as shown below:

Notice that \( f(x) \) and \( h(x) \) are both treated as if they are of length 4, and the circular convolution is also of length 4.
Linear Convolution as Circular Convolution

If $f(x)$ and $g(x)$ are both treated as if they are of length $4 + 2 - 1 = 5$, then the following circular convolution is calculated:

\begin{align*}
    g(0) &= f(4)h(1) + f(0)h(0) = 0 + 1 = 1 \\
    g(1) &= f(0)h(1) + f(1)h(0) = 1 + 3 = 4 \\
    g(2) &= f(1)h(1) + f(2)h(0) = 3 + -2 = 1 \\
    g(3) &= f(2)h(1) + f(3)h(0) = -2 + 1 = -1 \\
    g(4) &= f(3)h(1) + f(4)h(0) = 1 + 0 = 1
\end{align*}

This procedure is called “zero padding.” Notice that this circular convolution matches the linear convolution.

In general, if $f(x)$ has length $A$, and $h(x)$ has length $B$, and both $f(x)$ and $h(x)$ are zero padded out to length $C$, where $C \geq A + B - 1$, then the $C$-point circular convolution matches the linear convolution.
Figure 1: Linear convolution by the graphical method.
Figure 2: Circular convolution using the graphical method.