Optimal Quantization (Lloyd-Max)

\[ \int_{-\infty}^{\infty} (s - t_i) p(s) \, ds = 0 \quad s_i = \frac{t_i + t_{i+1}}{2} \]

- Minimize mean square quantization error: \( E[(s - q(s))^2] \)
- Consider only even PDF of data, \( p(s) = p(-s) \)
- Leads to odd quantizer function, \( q(s) = -q(-s) \)
- \( s_i \) – decision levels
- \( t_i \) – reconstruction levels
- Optimum decision levels lie halfway between the optimum reconstruction levels, which, in turn, lie at the centroid of the PDF in between the decision levels.

Image Transforms

- General forward transform:
  \[ T(u,v) = \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) \, g(x,y,u,v) \]
- General reverse transform:
  \[ f(u,v) = \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} T(x,y) \, h(x,y,u,v) \]
- Unitary transform if:
  \[ g(x,y,u,v) = h^*(x,y,u,v) \]

Transform Based Compression

Throwing out DFT coefficients

Original

3:1

4:1

Image Transforms

- \( g(x,y,u,v) \) form an orthonormal basis:
  \[ \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} g(x,y,u,v) \, g'(x,y,u',v') = \begin{cases} 1 & \text{if } u-u' \text{ and } v-v' \\ 0 & \text{otherwise} \end{cases} \]
- Separable transform if:
  \[ g(x,y,u,v) = g_1(x,u) \, g_2(y,v) \]
- Symmetric transform if:
  \[ g_1(x,u) = g_2(x,u) \]
Discrete Fourier Transform (DFT)

- DFT:
  \[ g(x, y, u, v) = W_N^u W_N^v \]
  \[ h(x, y, u, v) = \frac{1}{N^2} W_N^{-u} W_N^{-v} \]

- Unitary DFT:
  \[ g(x, y, u, v) = \frac{1}{N} W_N^u W_N^v \]
  \[ h(x, y, u, v) = \frac{1}{N} W_N^{-u} W_N^{-v} \]

Walsh-Hadamard Transform (WHT)

- Summation in modulo-2 arithmetic
- \( b_j(z) \) is the \( k \)th bit of \( z = (b, b, b, b, b, b, b, b, b) \)
- \( p_j(z) \):
  - convert \( z \) to Gray code
  - reverse bits of the Gray code
  - \( p_j(z) \) is the \( k \)th bit of the result

Walsh-Hadamard Transform (WHT)

\[ g(x, y, u, v) = h(x, y, u, v) - \alpha(u)\alpha(v) \cos \left( \frac{2\pi(u+1)v}{2N} \right) \cos \left( \frac{2\pi(u+1)v}{2N} \right) \]

\[ \alpha(u) = \begin{cases} \frac{1}{N} & \text{for } u = 0 \\ \frac{2}{N} & \text{for } u = 1, \ldots, N-1 \end{cases} \]

Discrete Cosine Transform (DCT)

- WHT and DCT are both fast transforms – FFT-like algorithms exist.
- DCT is not the real part of the DFT
Basis Images

\[
H_{uv} = \begin{bmatrix}
    h(0,0,u,v) & h(0,1,u,v) & \cdots & h(0,N-1,u,v) \\
    h(1,0,u,v) & h(1,1,u,v) & \cdots & h(1,N-1,u,v) \\
    \vdots & \vdots & \ddots & \vdots \\
    h(N-1,0,u,v) & h(N-1,1,u,v) & \cdots & h(N-1,N-1,u,v)
\end{bmatrix}
\]

\[
\delta(u-u_0, v-v_0) = \begin{cases} 
    1 & \text{if } u = u_0 \text{ and } v = v_0 \\
    0 & \text{otherwise}
\end{cases}
\]

\[
H_{uv} = \text{Trans}^{-1} \delta(u-u_0, v-v_0)
\]

Energy Compaction

- Different transforms achieve a different distribution of coefficient variance, \(\sigma^2_{T(u,v)}\), over transform space \((u,v)\).
- The Karhunen-Loève transform (KLT) has optimal energy compaction properties – KLT depends upon the image data.
- DCT usually is the best practical transform in terms of energy compaction and minimization of blocking artifacts.

Sub Image Size

- Image is subdivided into typical block sizes of 8x8 or 16x16 before transform encoding.
- Provides (some) gain in computational efficiency:
  \[
  \text{MN} \log_{2} \text{MN} \quad (mn \log_{2} mn) \quad \frac{\text{MN}}{mn} = MN \log_{2} mn
  \]
  \[
  \frac{\log_{2} \text{MN}}{\log_{2} mn} = \frac{\log_{2} 512}{\log_{2} 16} = \frac{9}{4} = 2.25
  \]
- Need to use FFT only in powers of 2.
- De-couples errors in different parts of the image.
- Disadvantages:
  - Better compression with larger sub images
  - Blocking artifacts.
Bit Allocation

- Bit allocation refers to the strategy used in keeping or discarding transform coefficients.
- Two types:
  - Zonal coding
  - Threshold coding – three types:
    - Single global threshold
    - N-largest coding
    - Variable/adaptive threshold

Zonal Coding

- Zonal mask:
  - Keep samples under mask
  - Same number of bits used to encode each sample
  - First normalize each sample by standard deviation, then use uniform quantization
- Zonal bit allocation:
  - Different number of bits allocated to different samples
  - Use non-uniform quantization (such as Lloyd-Max)

Threshold Coding

- Inherently adaptive
- Transform coefficients are re-ordered into a 1-d array
- Only coefficients greater than a threshold are retained
- Run-length encoding is used to take advantage of long strings of zeros corresponding to neglected coefficients
Threshold coding

- Single global threshold – one threshold used for all blocks in entire image.
- $N$-largest coding – only the $N$ largest coefficients in each block are retained.
- Coefficients are normalized before thresholding
  - Results in variable threshold
  - Threshold depends upon psychovisual importance of coefficient.

The JPEG Standard

- JPEG is not a single fixed format; there are several alternate formats with many options.
- Four modes of operation:
  - Sequential – lossy coding based on the DCT
  - Progressive – multiple scans; viewer can watch the image go from coarse to clear
  - Lossless – based on predictive coding
  - Hierarchical – multi-resolution; lower resolution versions are available before higher resolution versions
- Codec – encoder/decoder

JPEG Sequential Coding

- Middle gray level is shifted to zero by subtracting 128 from each pixel value.
- Output of DCT is 64 uniformly quantized coefficients
- Quantization:
  - $Q(u,v) - 8 \times 8$ quantization table containing integers from 1 to 255
  - Quantization table determined by application
  - Quantization table numbers express the visual significance of each DCT coefficient
  - DCT coefficients are divided by $Q(u,v)$, then rounded to nearest integer:
    \[
    \hat{C}_{uv} = \text{round}\left(\frac{C_{uv}}{Q(u,v)}\right)
    \]
  - Many of the quantized DCT coefficients will round down to zero.
JPEG Sequential Coding

- DC sample is coded separately; the difference from the previous block is coded.
- Remaining samples are coded in a zig-zag pattern
- Entropy coding
  - Huffman coding – Huffman code table determined by the application
  - Arithmetic coding – no tables required; statistical information optional

JPEG Sequential Decoding

- Entropy decoder
- Dequantizer
  \[ \hat{C}_c(u,v) = \hat{C}_c(u,v) Q(u,v) \]
- Inverse DCT
- Shift middle gray level back to 128
- Re-assemble blocks