EEL 6562
Image Processing & Computer Vision
Notes on Frequency Domain Theory

1 Fourier Transform of Continuous Functions

1.1 Definitions

Forward transform:

$$\boxed{F(u,v) \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}\,dx\,dy}$$

Reverse transform:

$$\boxed{f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}\,du\,dv}$$

$F(u,v)$ is, in general, a complex function, so it has a rectangular form:

$$F(u,v) = R(u,v) + jI(u,v)$$

where $R(u,v)$ and $I(u,v)$ are real functions of u and v. $F(u,v)$ also has a polar form:

$$F(u,v) = |F(u,v)|e^{-j\phi(u,v)}$$

where

$$|F(u,v)| = \left(R^2(u,v) + I^2(u,v)\right)^{\frac{1}{2}}$$

and

$$\phi(u,v) = \tan^{-1}\left(\frac{I(u,v)}{R(u,v)}\right)$$

Power spectrum:

$$P(u,v) = |F(u,v)|^2 = F(u,v)\,F^*(u,v) = R^2(u,v) + I^2(u,v)$$

1.2 Properties

Linearity:

$$\boxed{af(x,y) + bg(x,y) \overset{\mathcal{F}}{\longleftrightarrow} aF(u,v) + bG(u,v)}$$

Spatial translation:

$$\boxed{f(x-x_0, y-y_0) \overset{\mathcal{F}}{\longleftrightarrow} F(u,v)e^{-j2\pi(ux_0+vy_0)}}$$

Frequency translation:

$$\boxed{f(x,y)e^{j2\pi(u_0x+v_0y)} \overset{\mathcal{F}}{\longleftrightarrow} F(u-u_0, v-v_0)}$$

Differentiation:

$$\boxed{\frac{\partial^n}{\partial x^n} f(x,y) \overset{\mathcal{F}}{\longleftrightarrow} (j2\pi u)^n F(u,v)}$$

Proof:

$$\begin{aligned}
\frac{\partial^n}{\partial x^n} f(x,y) &= \frac{\partial^n}{\partial x^n}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}\,du\,dv\right\} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\left\{\frac{\partial^n}{\partial x^n} e^{j2\pi(ux+vy)}\right\}\,du\,dv \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\left\{(j2\pi u)^n e^{j2\pi(ux+vy)}\right\}\,du\,dv \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{(j2\pi u)^n F(u,v)\}\, e^{j2\pi(ux+vy)}\,du\,dv
\end{aligned}$$

$$\boxed{(-j2\pi x)^n f(x,y) \overset{\mathcal{F}}{\longleftrightarrow} \frac{\partial^n}{\partial u^n} F(u,v)}$$

Laplacian:

$$\boxed{\nabla^2 f(x,y) \overset{\mathcal{F}}{\longleftrightarrow} -(2\pi)^2(u^2+v^2)F(u,v)}$$

Scaling:

$$\boxed{f(ax,by) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|ab|}F(u/a, v/b)}$$

Separable functions:

$$\boxed{f_x(x)\,f_y(y) = f(x,y) \overset{\mathcal{F}}{\longleftrightarrow} F(u,v) = F_x(u)\,F_y(v)}$$

Proof:

$$\begin{aligned}
F(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_x(x)f_y(y)e^{-j2\pi(ux+vy)}\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{f_x(x)e^{-j2\pi ux}\right\}\left\{f_y(y)e^{-j2\pi vy}\right\}\,dx\,dy \\
&= \int_{-\infty}^{\infty} f_y(y)e^{-j2\pi vy}\left\{\int_{-\infty}^{\infty} f_x(x)e^{-j2\pi ux}\,dx\right\}dy \\
&= \left\{\int_{-\infty}^{\infty} f_x(x)e^{-j2\pi ux}\,dx\right\}\left\{\int_{-\infty}^{\infty} f_y(y)e^{-j2\pi vy}\,dy\right\} \\
&= F_x(u)F_y(v)
\end{aligned}$$

1.3 Convolution

Linear convolution in continuous spatial domain is defined as

$$\boxed{f(x,y) * g(x,y) \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s,t)g(x-s,y-t)\,ds\,dt}$$

Convolution is commutative and distributive over addition:

$$f(x,y) * g(x,y) = g(x,y) * f(x,y)$$

$$f(x,y) * (g(x,y) + h(x,y)) = f(x,y) * g(x,y) + f(x,y) * h(x,y)$$

The Dirac delta function is defined as the function $\delta(x,y)$ that satisfies:

$$\iint_R \delta(x,y)\,dA \triangleq \begin{cases} 1 & \text{if } (0,0) \in R; \\ 0 & \text{otherwise.} \end{cases}$$

The Dirac delta function is the identity element for convolution:

$$f(x,y) * \delta(x,y) = \delta(x,y) * f(x,y) = f(x,y)$$

The convolution theorem:

$$\boxed{f(x,y) * g(x,y) \overset{\mathcal{F}}{\longleftrightarrow} F(u,v)\,G(u,v)}$$

$$\boxed{f(x,y)\,g(x,y) \overset{\mathcal{F}}{\longleftrightarrow} F(u,v) * G(u,v)}$$

1.4 Common transform pairs

Some common Fourier transform pairs:

$$\boxed{\delta(x,y) \overset{\mathcal{F}}{\longleftrightarrow} 1}$$

$$\boxed{2\pi\sigma^2 e^{-2\pi^2\sigma^2(x^2+y^2)} \overset{\mathcal{F}}{\longleftrightarrow} e^{-(u^2+v^2)/(2\sigma^2)}}$$

Proof:

$$\begin{aligned} F(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2\pi\sigma^2 e^{-2\pi^2\sigma^2(x^2+y^2)} e^{-j2\pi(ux+vy)}\,dx\,dy \\ &= e^{-\frac{u^2+v^2}{2\sigma^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2\pi\sigma^2 e^{-2\pi^2\sigma^2(x^2+y^2)} e^{-j2\pi(ux+vy)} e^{\frac{u^2+v^2}{2\sigma^2}}\,dx\,dy \\ &= e^{-\frac{u^2+v^2}{2\sigma^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2\pi\sigma^2 e^{-2\pi^2\sigma^2x^2 - j2\pi ux + \frac{u^2}{2\sigma^2}} \\ &\qquad\qquad \cdot e^{-2\pi^2\sigma^2y^2 - j2\pi vy + \frac{v^2}{2\sigma^2}}\,dx\,dy \end{aligned}$$

Working with the first exponent after the integral signs:

$$\begin{aligned} -2\pi^2\sigma^2x^2 - j2\pi ux + \frac{1}{2\sigma^2}u^2 &= -\frac{1}{2\sigma^2}\left(4\pi^2\sigma^4x^2 + j4\pi\sigma^2ux + u^2\right) \\ &= -\frac{1}{2\sigma^2}\left(2\pi\sigma^2x + ju\right)^2 \end{aligned}$$

The second exponent after the integral signs is similarly manipulated:

$$F(u,v) = e^{-\frac{u^2+v^2}{2\sigma^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2\pi\sigma^2 e^{-\frac{1}{2\sigma^2}\left(2\pi\sigma^2 x + ju\right)^2} e^{-\frac{1}{2\sigma^2}\left(2\pi\sigma^2 y + jv\right)^2}\, dx\, dy$$

Perform the following substitution:

$$\begin{aligned} r &= 2\pi\sigma^2 x + ju & s &= 2\pi\sigma^2 y + jv \\ dr &= 2\pi\sigma^2 dx & ds &= 2\pi\sigma^2 dy \end{aligned}$$

$$F(u,v) = e^{-\frac{u^2+v^2}{2\sigma^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} e^{-\frac{r^2+s^2}{2\sigma^2}}\, dr\, ds$$

The integral of a two dimensional Gaussian distribution is exactly unity:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} e^{-\frac{r^2+s^2}{2\sigma^2}}\, dr\, ds = 1$$

$$F(u,v) = e^{-\frac{u^2+v^2}{2\sigma^2}}$$

$$\boxed{\mathrm{Rect}_{a,b}(x,y) \overset{\mathcal{F}}{\longleftrightarrow} ab\,\frac{\sin \pi ua}{\pi ua}\,\frac{\sin \pi vb}{\pi vb}}$$

where

$$\mathrm{Rect}_{a,b}(x,y) \triangleq \begin{cases} 1 & \text{if } |x| \le a/2 \text{ and } |y| \le b/2; \\ 0 & \text{otherwise.} \end{cases}$$

$$\boxed{\cos(2\pi u_0 x + 2\pi v_0 y) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\left(\delta(u+u_0, v+v_0) + \delta(u-u_0, v-v_0)\right)}$$

$$\boxed{\sin(2\pi u_0 x + 2\pi v_0 y) \overset{\mathcal{F}}{\longleftrightarrow} j\frac{1}{2}\left(\delta(u+u_0, v+v_0) - \delta(u-u_0, v-v_0)\right)}$$

2 Fourier Transform of Discrete Functions

2.1 Definitions

For functions that are either discrete and finite or discrete and periodic, we use the discrete Fourier transform (DFT):

$$\boxed{F(u,v) \triangleq \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)}}$$

Inverse transform:

$$\boxed{f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v) e^{j2\pi(ux/M + vy/N)}}$$

Define:

$$W_N \triangleq e^{-j\frac{2\pi}{N}}$$

Forward transform:

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\, W_N^{ux}\, W_M^{vy}$$

Inverse transform:

$$f(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v)\, W_N^{-ux}\, W_M^{-vy}$$

$F(u,v)$ is, in general, a complex function, so it has a rectangular form:

$$F(u,v) = R(u,v) + jI(u,v)$$

where $R(u,v)$ and $I(u,v)$ are real functions of u and v. $F(u,v)$ also has a polar form:

$$F(u,v) = |F(u,v)|e^{-j\phi(u,v)}$$

where

$$|F(u,v)| = \left(R^2(u,v) + I^2(u,v)\right)^{\frac{1}{2}}$$

and

$$\phi(u,v) = \tan^{-1}\left(\frac{I(u,v)}{R(u,v)}\right)$$

2.2 Properties

Power spectrum:

$$P(u,v) = |F(u,v)|^2 = F(u,v)\, F^*(u,v) = R^2(u,v) + I^2(u,v)$$

Periodicity:

$$F(u,v) = F(u+pM, v+qN) \text{ for all integers } p \text{ and } q$$

DFT of the complex conjugate:

$$\boxed{f^*(x,y) \overset{\mathcal{DFT}}{\longleftrightarrow} F^*(-u,-v)}$$

The above DFT pair implies that for real $f(x,y)$, the DFT is conjugate symmetric:

$$\boxed{F(u,v) = F^*(-u,-v)}$$

Conjugation in frequency domain:

$$\boxed{f^*(-x,-y) \overset{\mathcal{DFT}}{\longleftrightarrow} F^*(u,v)}$$

DC term:

$$F(0,0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)$$

Also:

$$F(M/2, N/2) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\,(-1)^{x+y}$$

$$F(M/2, 0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\,(-1)^{x}$$

$$F(0, N/2) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\,(-1)^{y}$$

Linearity:

$$\boxed{af(x,y) + bg(x,y) \overset{\mathcal{DFT}}{\longleftrightarrow} aF(u,v) + bG(u,v)}$$

So far, we have tacitly assumed that $f(x,y)$ is a spatially finite image and that its region of support is $0 \le x \le M-1$, $0 \le y \le N-1$. Define the periodic extension of $f(x,y)$:

$$\tilde{f}(x,y) \triangleq f(x \bmod M, y \bmod N)$$

The DFT has a dual interpretation:

- The DFT consists of frequency samples of the continuous Fourier transform of a finite image.
- The DFT corresponds to the Fourier series expansion of a periodic signal.

Both interpretations are useful, and we can switch back and forth as long as we are careful.

Spatial translation:

$$\boxed{\tilde{f}(x - x_0, y - y_0) \overset{\mathcal{DFT}}{\longleftrightarrow} F(u,v)\,W_M^{ux_0}\,W_N^{vy_0}}$$

Frequency translation:

$$\boxed{f(x,y)\,W_M^{-u_0 x}\,W_N^{-v_0 y} \overset{\mathcal{DFT}}{\longleftrightarrow} F(u - u_0, v - v_0)}$$

Proof:

$$F(r,s) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\,W_N^{rx}\,W_M^{sy}$$

Let $r = u - u_0$ and $s = v - v_0$:

$$\begin{aligned}
F(u - u_0, v - v_0) &= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\,W_N^{(u-u_0)x}\,W_M^{(v-v_0)y} \\
&= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\,W_N^{ux} W_N^{-u_0 x}\,W_M^{vy} W_M^{-v_0 y} \\
&= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left\{ f(x,y)\,W_N^{-u_0 x}\,W_M^{-v_0 y} \right\}\,W_N^{ux}\,W_M^{vy}
\end{aligned}$$

A special case of frequency translation:

$$\boxed{f(x,y)\,(-1)^{x+y} \overset{\mathcal{DFT}}{\longleftrightarrow} F(u-M/2, v-N/2)}$$

Separable functions:

$$\boxed{f_x(x)\,f_y(y) = f(x,y) \overset{\mathcal{DFT}}{\longleftrightarrow} F(u,v) = F_x(u)\,F_y(v)}$$

2.3 Convolution and correlation

Linear discrete convolution:

$$\boxed{f(x,y) * g(x,y) \triangleq \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} f(s,t)\,g(x-s, y-t)}$$

Circular discrete convolution:

$$\boxed{f(x,y) \circledast g(x,y) = \sum_{s=0}^{M-1} \sum_{t=0}^{N-1} \tilde{f}(s,t)\,\tilde{g}(x-s, y-t)}$$

Both forms of convolution defined above are commutative and distribute over addition:

$$f(x,y) * g(x,y) = g(x,y) * f(x,y)$$

$$f(x,y) * (g(x,y) + h(x,y)) = f(x,y) * g(x,y) + f(x,y) * h(x,y)$$

and

$$f(x,y) \circledast g(x,y) = g(x,y) \circledast f(x,y)$$

$$f(x,y) \circledast (g(x,y) + h(x,y)) = f(x,y) \circledast g(x,y) + f(x,y) \circledast h(x,y)$$

The unit sample function, $\delta(x,y)$, is defined as:

$$\delta(x,y) \triangleq \begin{cases} 1 & \text{if } x = y = 0; \\ 0 & \text{otherwise.} \end{cases}$$

The unit sample function is the identity element for both forms of discrete convolution:

$$f(x,y) * \delta(x,y) = \delta(x,y) * f(x,y) = f(x,y)$$

and

$$f(x,y) \circledast \delta(x,y) = \delta(x,y) \circledast f(x,y) = f(x,y)$$

The discrete form of the convolution theorem applies only to circular convolution:

$$\boxed{f(x,y) \circledast g(x,y) \overset{\mathcal{DFT}}{\longleftrightarrow} F(u,v)\,G(u,v)}$$

$$\boxed{f(x,y)\,g(x,y) \overset{\mathcal{DFT}}{\longleftrightarrow} \frac{1}{MN}\,F(u,v) \circledast G(u,v)}$$

However, the theorem given below gives us a way to calculate the linear convolution using circular convolution:

Suppose $f(x, y)$ is $A \times B$ and $h(x, y)$ is $C \times D$. Suppose further that we zero pad $f(x, y)$ and $h(x, y)$ out to $P \times Q$ and call the new functions $f_e(x, y)$ and $h_e(x, y)$, respectively. Then if $P \geq A + C - 1$ and $Q \geq B + D - 1$, the results of circular and linear convolution match:

$$f_e(x, y) * g_e(x, y) = f_e(x, y) \circledast g_e(x, y) \text{ for } 0 \leq x \leq P - 1 \text{ and } 0 \leq y \leq Q - 1.$$

Correlation:

$$\boxed{f(x, y) \circ g(x, y) \triangleq \sum_{s=0}^{M-1} \sum_{t=0}^{N-1} f^*(s, t)\, g(x + s, y + t)}$$

Relationship between convolution and correlation:

$$f(x, y) \circ g(x, y) = \frac{1}{MN} f^*(-x, -y) * g(x, y)$$

If $f(x, y)$ and $g(x, y)$ are sufficiently zero padded so that linear and circular convolution correspond, then

$$\boxed{f(x, y) \circ g(x, y) \overset{\mathcal{DFT}}{\longleftrightarrow} \frac{1}{MN} F^*(u, v)\, G(u, v),}$$

$$\boxed{f^*(x, y)\, g(x, y) \overset{\mathcal{DFT}}{\longleftrightarrow} F(u, v) \circ G(u, v).}$$

2.4 Example 1

Find the frequency response of the filter with the following mask:

1	0	0
0	0	0
0	0	1

$$g(x, y) = f(x + 1, y + 1) + f(x - 1, y - 1)$$
$$G(u, v) = F(u, v) W_M^{-u} W_N^{-v} + F(u, v) W_M^{u} W_N^{v}$$

Frequency response:

$$\begin{aligned} H(u, v) &= \frac{G(u, v)}{F(u, v)} \\ &= W_M^{-u} W_N^{-v} + W_M^{u} W_N^{v} \\ &= e^{j2\pi(u/M + v/N)} + e^{-j2\pi(u/M + v/N)} \\ &= 2\cos\left(2\pi(u/M + v/N)\right) \end{aligned}$$

Magnitude and phase of frequency response:

$$|H(u, v)| = 2|\cos\left(2\pi(u/M + v/N)\right)|$$

$$\phi(u, v) = 0$$

2.5 Example 2

Find the frequency response of the filter with the following mask:

-1	0	0
0	0	0
0	0	1

$$g(x,y) = f(x+1, y+1) - f(x-1, y-1)$$
$$G(u,v) = F(u,v)W_M^{-u}W_N^{-v} - F(u,v)W_M^{u}W_N^{v}$$

Frequency response:

$$\begin{aligned} H(u,v) &= \frac{G(u,v)}{F(u,v)} \\ &= W_M^{-u}W_N^{-v} - W_M^{u}W_N^{v} \\ &= e^{j2\pi(u/M+v/N)} - e^{-j2\pi(u/M+v/N)} \\ &= j2\sin\left(2\pi(u/M+v/N)\right) \end{aligned}$$

Magnitude and phase of frequency response:

$$|H(u,v)| = 2|\sin\left(2\pi(u/M+v/N)\right)|$$

$$\phi(u,v) = \begin{cases} \pi/2 & \text{if } \sin\left(2\pi(u/M+v/N)\right) > 0 \\ -\pi/2 & \text{if } \sin\left(2\pi(u/M+v/N)\right) < 0 \end{cases}$$

3 Relationship Between Discrete and Continuous Fourier Transforms

Define the following transform pairs:

$$f_c(x,y) \xleftrightarrow{\mathcal{F}} F_c(u_c, v_c)$$

$$f(x,y) \xleftrightarrow{\mathcal{DFT}} F(u,v)$$

To simplify this discussion, let us assume that the sampling rate is exactly unity; i.e. $f_s = T_s = 1$. Suppose $f_c(x,y)$ has a finite region of support; specifically, suppose $f_c(x,y) = 0$ for $x \leq -1$, $x \geq M$, $y \leq -1$, and $y \geq M$.

Let $f(x,y) = f_c(x,y)$ for integers x and y in the intervals $0 \leq x \leq M-1$ and $0 \leq y \leq N-1$. Then

$$F(u,v) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} F_c(u/M + k, v/N + l).$$

Suppose $F_c(u,v) \approx 0$ for $|u| > 1/2$, and $|v| > 1/2$, then for integers u and v in the intervals $-M/2 < u < M/2$ and $-N/2 < v < N/2$,

$$F(u,v) \approx F_c(u/M, v/N).$$

Since a function cannot be both spatially finite and band limited, the above expression can only be approximately satisfied.