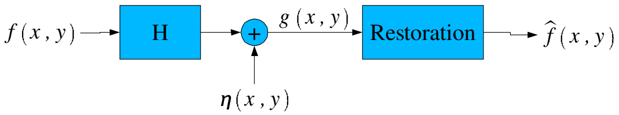
Image Degradation Model



Very general form of degradation function:

$$g(x,y)=H[f(x,y)]+\eta(x,y)$$

Linear Degradation Function

If H is a linear operator

$$H[af_1(x,y)+bf_2(x,y)]=aH[f_1(x,y)]+bH[f_2(x,y)]$$

then H can be expressed as a superposition integral of the first kind

$$H[f(x,y)]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)h(x,\alpha,y,\beta)\,d\alpha\,d\beta$$

where the impulse response h is defined as

$$h(x,\alpha,y,\beta)=H[\delta(x-\alpha,y-\beta)]$$

- Impulse response
- Point Spread Function (PSF)

Position Invariant Degradation Function

If H is position invariant

$$g(x,y)=H(f(x,y)) \Rightarrow g(x-\alpha,y-\beta)=H(f(x-\alpha,y-\beta)) \quad \forall\alpha,\beta\in\mathbb{R}$$

then H can be expressed as a convolution integral

$$H[f(x,y)]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)h(x-\alpha,y-\beta)\,d\alpha\,d\beta$$

where the position invariant impulse response h is defined as

$$h(x,y)=H[\delta(x,y)]$$

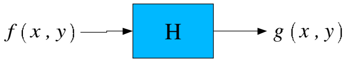
Then the degradation model becomes

$$g(x,y)=h(x,y)*f(x,y)+\eta(x,y)$$

or, in frequency domain

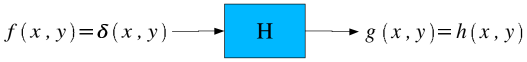
$$G(u,v)=H(u,v)F(u,v)+N(u,v)$$

Estimating the Degradation Function



For a known input and output (neglecting noise):

$$H(u,v)=\frac{G(u,v)}{F(u,v)}$$



Estimation by Modeling

Blurring due to atmospheric turbulence:

$$H(u,v)=e^{-k(u^2+v^2)^{5/6}}$$

Blurring due to lens defocus:

$$H(u,v)=\frac{J_1\left(a\sqrt{u^2+v^2}\right)}{a\sqrt{u^2+v^2}}$$

J_1 is a Bessel function of the first kind, first order

$$J_1(x)=\frac{x}{2}-\frac{x^3}{2^2\cdot 4}+\frac{x^5}{2^2\cdot 4^2\cdot 6}-\frac{x^7}{2^2\cdot 4^2\cdot 6^2\cdot 8}+\cdots$$

Camera Motion

Very general model of camera motion:

$$g(x,y)=\int_{-T/2}^{T/2} f(x-x_0(t),y-y_0(t))\,dt$$

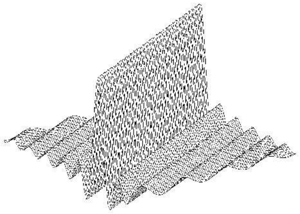
$$G(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{-T/2}^{T/2} f(x-x_0(t),y-y_0(t))\,dt\right\}e^{-j2\pi(ux+vy)}\,dx\,dy$$

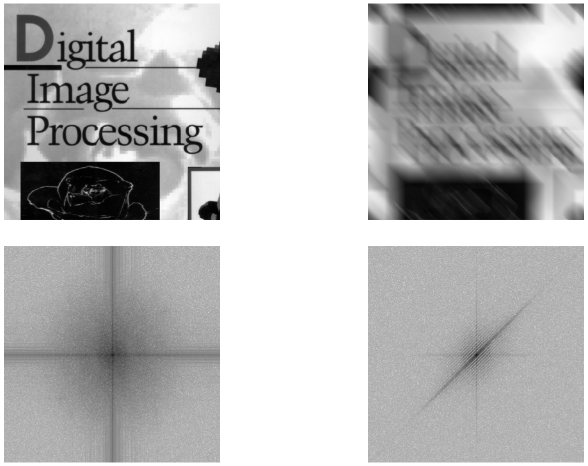
$$G(u,v)=\int_{-T/2}^{T/2}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-x_0(t),y-y_0(t))e^{-j2\pi(ux+vy)}\,dx\,dy\right\}dt$$

Camera Motion

Linear camera motion:

$$x_0(t)=\frac{at}{T} \qquad y_0(t)=\frac{bt}{T}$$





Inverse Filtering

$$G(u,v)=F(u,v)\,H(u,v)$$

$$\widehat{F}(u,v)=\frac{G(u,v)}{H(u,v)}$$

1. Assumes no noise.
2. Problem if *H(u,v)* has a zero.

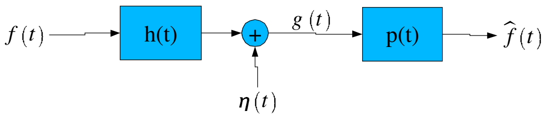
What happens to the noise?

$$G(u,v)=F(u,v)\,H(u,v)+N(u,v)$$

$$\widehat{F}(u,v)=\frac{G(u,v)}{H(u,v)}=F(u,v)+\frac{N(u,v)}{H(u,v)}$$

Pseudo-inverse filter:

1-d Wiener Filter



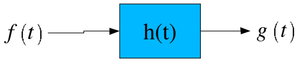
We seek a filter *p*(*t*) such that:

$$\hat{f}(t)=g(t)*p(t)$$

is an estimate of *f*(*t*) that minimizes:

$$e^2=E\left[\left(f(t)-\hat{f}(t)\right)^2\right]$$

Stochastic Inputs to Linear Systems



$$g(t)=f(t)*h(t)$$

$$R_{ff}(\tau)=E[f(t+\tau)f^*(t)]$$

$$R_{fg}(\tau)=E[f(t+\tau)g^*(t)]$$

1-d Wiener Filter

Assume estimation error is orthogonal to measurements for all time lags:

$$E\left[\left(f(t)-\hat{f}(t)\right)g^*(t-\tau)\right]=0$$

$$E\left[\left(f(t)-\int_{-\infty}^{\infty}p(\alpha)g(t-\alpha)\,d\alpha\right)g^*(t-\tau)\right]=0$$

$$E[f(t)g^*(t-\tau)]=\int_{-\infty}^{\infty}p(\alpha)E[g(t-\alpha)g^*(t-\tau)]\,d\alpha$$

1-d Wiener Filter

- Assumptions:
  - Both the uncorrupted signal & noise signal are stochastic
  - We know the autocorrelation functions of the uncorrupted signal & noise
  - We know the degradation function
  - The measurement data is known for all time
- The Wiener filter minimizes the expected value of the squared estimation error
- Under these assumptions, the Wiener filter is the optimal linear, time invariant filter to estimate the uncorrpted signal

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2-d Wiener Filter

$$\widehat{F}(u,v)=\left(\frac{S_{ff}(u,v)H^{*}(u,v)}{s_{ff}(u,v)|H(u,v)|^{2}+S_{\eta\eta}(u,v)}\right)G(u,v)$$

$$\widehat{F}(u,v)=\left(\frac{1}{H(u,v)}\cdot\frac{|H(u,v)|^{2}}{|H(u,v)|^{2}+\dfrac{S_{\eta\eta}(u,v)}{S_{ff}(u,v)}}\right)G(u,v)$$

- A zero in H(u,v) is not a problem (unless the noise spectral density just happens to have a zero in exactly the same location)
- Problem: image spectral density and noise spectral density difficult to know exactly.

2-d Wiener Filter

If we assume:

$$\frac{S_{\eta\eta}(u,v)}{S_{ff}(u,v)}=K$$

then

$$\widehat{F}(u,v)=\left(\frac{1}{H(u,v)}\cdot\frac{|H(u,v)|^{2}}{|H(u,v)|^{2}+K}\right)G(u,v)$$